Tensor Norms for Quantum Entanglement

(Part 2)

Keywords: Quantum Entanglement, Tensor Norms, Separability, Nuclear Norm, Partial Transpose, PPT Criterion, Unextendible Product Basis

Abstract

We present the Positive Partial Transpose (PPT) criterion as a powerful, computable test for entanglement. Furthermore, we explore its limitations by discussing PPT-entangled states, constructed using the notion of an Unextendible Product Basis (UPB), which necessitates even more general criteria for a complete characterization of entanglement.

1 Recap

We begin by recalling the basic definitions of states in finite-dimensional quantum mechanics.

1.1 Definition (Quantum State). A **quantum state** ρ is a positive semi-definite, self-adjoint operator on a complex Hilbert space $\mathcal{H}$ with unit trace. For these notes, we consider a finite d-dimensional space, $\mathcal{H} = \mathbb{C}^d$.

1.2 Definition (Pure State). A **pure state** is a state of rank one. It can be written as a projection operator $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle \in \mathbb{C}^d$ is a unit vector, i.e., $\langle\psi|\psi\rangle = 1$.

1.3 Definition (Mixed State). A **mixed state** is a state that is not pure (i.e., has rank greater than one). Any mixed state can be expressed as a convex combination of pure states:

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \quad \text{with} \quad \lambda_i > 0, \sum_i \lambda_i = 1$$

1.4 Definition (Separable Pure State). A **separable pure state** in a k-partite Hilbert space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_k}$ is a unit vector $|\psi\rangle$ that can be written as a tensor product:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_k\rangle, \quad |\psi_j\rangle \in \mathbb{C}^{d_j}$$

That is, $|\psi\rangle$ is separable if and only if it is a product vector across all k subsystems.

1.5 Definition (Separable Mixed State). A **separable mixed state** (density operator) ρ on $\mathcal{H} = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_k}$ is a convex combination of product states of density operators:

$$\rho = \sum_i \lambda_i \left(\rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(k)} \right)$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, and each $\rho_i^{(j)}$ is a density operator on $\mathbb{C}^{d_j}$.
Remark: By Carathéodory's theorem, the $\rho_i^{(j)}$ can always be chosen to be pure states, i.e., $\rho_i^{(j)} = |\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|$ for some unit vector $|\psi_i^{(j)}\rangle \in \mathbb{C}^{d_j}$.

When dealing with a composite system, the total Hilbert space is a tensor product of the individual spaces, e.g., $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. This structure gives rise to the phenomenon of entanglement.

A state that is not separable is called **entangled**. Detecting entanglement is a central problem in quantum information theory.

2 The PPT Criterion for Entanglement

One of the most practical tools for detecting entanglement is the **nuclear norm criterion**, which is equivalent to the famous Positive Partial Transpose (PPT) criterion. We present the nuclear norm test first, then explain its equivalence to the positivity condition.

2.1 Theorem (Entanglement Test via Nuclear Norm). *Let ρ be a bipartite density operator on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. If the nuclear norm of its partial transpose is greater than one,*

$$\|\rho^{\Gamma_B}\|_* > 1$$

then the state ρ is entangled.

Proof. Suppose ρ is a separable bipartite density operator:

$$\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B$$

where $p_k \geq 0$, $\sum_k p_k = 1$, and ρ_k^A, ρ_k^B are density operators.

The partial transpose with respect to B is:

$$\rho^{\Gamma_B} = \sum_k p_k \rho_k^A \otimes (\rho_k^B)^T$$

Each $(\rho_k^B)^T$ is also a density operator (since the transpose preserves positivity and trace), so ρ^{Γ_B} is a convex combination of product density operators, and thus a density operator itself (i.e., positive semi-definite and $\mathrm{Tr}(\rho^{\Gamma_B}) = 1$).

To compute the nuclear norm, recall that for any $S \in \mathcal{M}_d(\mathbb{C})$, the singular value decomposition (SVD) is $S = \sum_i \sigma_i |u_i\rangle\langle v_i|$, where $\sigma_i \geq 0$ are the singular values. For positive semi-definite operators, the SVD coincides with the spectral decomposition: $S = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ with $\lambda_j \geq 0$.

Thus, for any positive semi-definite A, $\|A\|_* = \sum_j \lambda_j = \mathrm{Tr}(A)$.

Therefore, for separable ρ,

$$\|\rho^{\Gamma_B}\|_* = \mathrm{Tr}(\rho^{\Gamma_B}) = 1$$

Summary of steps:

- Partial transpose of a separable state is a convex combination of density operators $\implies$ density operator (positive semi-definite, trace 1).
- For positive semi-definite operators, nuclear norm equals trace.
- Thus, $\|\rho^{\Gamma_B}\|_* = 1$ for separable ρ.

Contrapositive: If $\|\rho^{\Gamma_B}\|_* > 1$, then ρ cannot be separable, i.e., ρ is entangled. □

Remark (Relation to PPT criterion):

- **Really used:** If ρ is separable, then ρ^{Γ_B} is also a density operator.
- **Can show:** For any state, $(\rho^{\Gamma_B})^* = \rho^{\Gamma_B}$ and $\mathrm{Tr}(\rho^{\Gamma_B}) = 1$ (i.e., ρ^{Γ_B} is always Hermitian and trace 1).
- If ρ is separable, then ρ^{Γ_B} is also positive semi-definite.

Positive Partial Transpose (PPT) Test: If $\rho^{\Gamma_B} \not\geq 0$, then ρ is entangled.

3 Limits of the PPT Criterion: PPT-Entangled States

The PPT criterion is a powerful tool, but it is not a perfect one. It is a necessary condition for separability, but it is not sufficient in general. There exist entangled states that nonetheless have a positive partial transpose. These are known as **PPT-entangled states** or **bound entangled states**. A common way to construct such states is by using an Unextendible Product Basis (UPB).

3.1 Definition (Unextendible Product Basis). An **Unextendible Product Basis (UPB)** is a set of orthonormal product vectors $\{|x_i\rangle = |a_i\rangle \otimes |b_i\rangle\}$ that span a proper subspace $\mathcal{E} \subset \mathcal{H}_A \otimes \mathcal{H}_B$, with the property that there is no other product vector $|\psi\rangle = |a\rangle \otimes |b\rangle$ that is orthogonal to every vector in the set. In other words, the orthogonal complement $\mathcal{E}^\perp$ contains no product vectors.

Using a UPB, we can construct a class of PPT-entangled states.

3.2 Proposition (Werner-like states from UPBs). *Let $\{|x_i\rangle\}_{i=1}^k$ be a UPB in a d-dimensional space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $P_\mathcal{E} = \sum_{i=1}^k |x_i\rangle\langle x_i|$ be the projector onto the subspace spanned by the UPB. The state defined by*

$$\rho_\mathcal{E} = \frac{1}{d-k}(\mathrm{Id} - P_\mathcal{E})$$

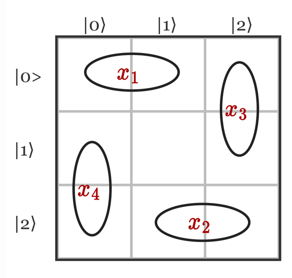
is a PPT-entangled state (i.e., it is entangled but has a positive partial transpose).

Proof Sketch.

- **PPT Property:** The partial transpose of $\rho_\mathcal{E}$ is $\rho_\mathcal{E}^{\Gamma_B} = \frac{1}{d-k}(\mathrm{Id} - P_\mathcal{E}^{\Gamma_B})$. The partial transpose of the projector is $P_\mathcal{E}^{\Gamma_B} = \sum_i (|a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|)^{\Gamma_B} = \sum_i |a_i\rangle\langle a_i| \otimes |\bar{b_i}\rangle\langle\bar{b_i}|$. Since $\{|x_i\rangle\}$ are orthonormal, the set $\{|a_i\rangle \otimes |\bar{b_i}\rangle\}$ is also an orthonormal set of product vectors. Thus, $P_\mathcal{E}^{\Gamma_B}$ is also a projector. This means $\rho_\mathcal{E}^{\Gamma_B}$ is proportional to a projector $(\mathrm{Id} - P_\mathcal{E}^{\Gamma_B})$ and is therefore positive semi-definite. So, the state is PPT.
- **Entanglement:** The **support** of a density operator ρ, denoted $\mathrm{supp}(\rho)$, is the orthogonal complement of its kernel.
If a state is separable, its support is spanned by product vectors. The support of $\rho_\mathcal{E}$ is the subspace $\mathcal{E}^\perp$. By the definition of a UPB, the subspace $\mathcal{E}^\perp$ contains no product vectors. Therefore, $\rho_\mathcal{E}$ cannot be separable.

□

3.1 Example: The "Tiles" UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$



A well-known example of a UPB is the "Tiles" basis, which consists of 5 vectors in a 9-dimensional space:

- $|x_1\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle - |1\rangle)$
- $|x_2\rangle = \frac{1}{\sqrt{2}}|2\rangle \otimes (|1\rangle - |2\rangle)$
- $|x_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |2\rangle$
- $|x_4\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \otimes |0\rangle$
- $|x_5\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle) \otimes (|0\rangle + |1\rangle + |2\rangle)$

Claim: The set $\{|x_i\rangle\}_{i=1}^5$ is unextendible.
Suppose, for contradiction, that there exists a product vector $|\psi\rangle = |a\rangle \otimes |b\rangle$ such that $\langle x_i|\psi\rangle = 0$ for all $i = 1, \ldots, 5$.
For each i, this gives a linear equation in the components of $|a\rangle$ and $|b\rangle$.
Explicitly, for $|x_1\rangle = |0\rangle \otimes (|0\rangle - |1\rangle)$, we have:

$$\langle x_1|\psi\rangle = \langle 0|a\rangle \cdot (\langle 0|b\rangle - \langle 1|b\rangle) = 0$$

and similarly for the other $|x_i\rangle$. Each equation implies an orthogonality condition for $|a\rangle$ or $|b\rangle$
There are 5 equations. Hence by pigeonhole principle, this forces $|a\rangle$ or $|b\rangle$ to be orthogonal to at least 3 independent vectors in $\mathbb{C}^3$, which is only possible if $|a\rangle = 0$ or $|b\rangle = 0$.
Thus, there is no nonzero product vector orthogonal to all $|x_i\rangle$.
Therefore, the set is unextendible.

4 Toward a General Criterion for Entanglement

The existence of PPT-entangled states shows that the PPT criterion is not a complete solution to the entanglement detection problem. The ultimate goal is to find a condition that is both necessary and sufficient for separability. This leads us to a more geometric and functional-analytic perspective.

4.1 The Geometry of Separable States

The set of all quantum states $\mathcal{D}$ is a convex set. The subset of separable states, which we denote by S, is also a convex set. Specifically, S is the convex hull of all pure product states.

$$S = \mathrm{conv}\{|\psi\rangle\langle\psi| : |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes \ldots\}$$

A state ρ is separable if and only if $\rho \in S$. All other states in $\mathcal{D} \setminus S$ are entangled. This geometric picture suggests that we can test for entanglement by determining if a state lies inside or outside the convex set S.

Functional analysis provides a tool for this: the Minkowski functional (or gauge function) of a convex set.

4.1 Definition (Minkowski Functional). For a convex set K containing the origin, its Minkowski functional $p_K(x)$ is defined as:

$$p_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}$$

Under suitable conditions (e.g., K is closed and bounded), the set K can be recovered from its functional as $K = \{x : p_K(x) \leq 1\}$.

Applying this to our problem, if we can define a functional $p_S(\rho)$ for the set of separable states S, then a state ρ would be separable if and only if $p_S(\rho) \leq 1$. This would provide the desired "if and only if" criterion.

4.2 Theorem (Evaluation of Minkowski Functional for Pure Tensors). *Let $K = \mathrm{conv}\{x^{(1)} \otimes \cdots \otimes x^{(k)} : \|x^{(l)}\| = 1\}$ be the convex hull of unit-norm pure tensors in $\mathcal{H}$. Then, for any $\rho \in \mathcal{H}$,*

$$p_K(\rho) = \inf\left\{ \sum_j \prod_{l=1}^k \|y_j^{(l)}\| : \rho = \sum_j y_j^{(1)} \otimes \cdots \otimes y_j^{(k)} \right\}$$

4.3 Definition (Projective Tensor Norm). Let $\mathcal{H} = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_k}$. For $\rho \in \mathcal{H}$, the **projective tensor norm** is defined as

$$\|\rho\|_\pi = \inf\left\{ \sum_j \prod_{l=1}^k \|y_j^{(l)}\| : \rho = \sum_j y_j^{(1)} \otimes \cdots \otimes y_j^{(k)} \right\}$$

where the infimum is over all possible decompositions of ρ as a sum of pure tensors $y_j^{(1)} \otimes \cdots \otimes y_j^{(k)}$.

While this provides a complete theoretical answer, computing the projective norm is an NP-hard problem in general, making it impractical for direct application. The search for computable and powerful entanglement criteria remains an active area of research.