Digression: The History of Tensor Norms

Tensor Norms for Quantum Entanglement

Recap: Norm Criterion for Separability

1. Setup

- Fix a k-partite, $d_1 \times \cdots \times d_k$ Hilbert space:

$$\mathcal{H} = \bigotimes_{j=1}^{k} \mathbb{C}^{d_j}$$

- A *(normalised) density operator* $\rho \in \bigotimes_{j=1}^{k} M_{d_j}(\mathbb{C})$ satisfies:

$$\rho \geq 0 \quad \text{and} \quad \operatorname{Tr} \rho = 1$$

The Projective Norm

For a density operator $\rho \in \bigotimes_{j=1}^{k} M_{d_j}(\mathbb{C})$

The projective norm is defined over the L2 norms of these underlying vectors:

$$\|\rho\|_\pi = \inf \left\{ \sum_i \prod_{j=1}^{k} \|\phi_i^{(j)}\|_2 \, \|\psi_i^{(j)}\|_2 : \rho = \sum_i \bigotimes_{j=1}^{k} |\phi_i^{(j)}\rangle\langle\psi_i^{(j)}| \right\}$$

The Key Insight: The trace norm (or nuclear norm, $\|\cdot\|_1$) of a rank-one operator is precisely the product of the L2 norms of its constituent vectors.

$$\big\| |\phi\rangle\langle\psi| \big\|_1 = \|\phi\|_2 \|\psi\|_2$$

By substituting this identity, the vector-based definition transforms into one based on the nuclear norms of the component matrices.

2. The Projective Norm (Rewritten)

For a simple tensor $X_1 \otimes \cdots \otimes X_k \in \bigotimes_{j=1}^{k} M_{d_j}(\mathbb{C})$, we define:

$$\|X_1 \otimes \cdots \otimes X_k\|_\pi := \prod_{j=1}^{k} \|X_j\|_1, \quad \text{where } \|\cdot\|_1 \text{ is the trace norm.}$$

We extend this by infimum to *all* tensors ρ in the space:

$$\|\rho\|_\pi := \inf \left\{ \sum_i \|Y_i^{(1)}\|_1 \cdots \|Y_i^{(k)}\|_1 \; : \; \rho = \sum_i Y_i^{(1)} \otimes \cdots \otimes Y_i^{(k)} \right\}$$

3. The Separability Criterion

A density operator ρ is **separable** if and only if its projective tensor norm is at most 1.

$$\rho \text{ separable} \quad \Longleftrightarrow \quad \|\rho\|_\pi \leq 1$$

(Equivalently, $\|\rho\|_\pi > 1 \iff \rho$ is entangled).

Proof ($\Rightarrow$): separable $\Longrightarrow \|\rho\|_\pi \leq 1$

1. By definition, a separable state can be written as a convex combination:
$$\rho = \sum_i \lambda_i \, \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(k)}$$
with $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, and each $\rho_i^{(j)}$ being a single-system density operator (i.e., $\rho_i^{(j)} \geq 0$, $\operatorname{Tr} \rho_i^{(j)} = 1$).
2. This is already a valid decomposition. Using the definition of $\|\cdot\|_\pi$, this specific decomposition gives an upper bound:
$$\|\rho\|_\pi \leq \sum_i \prod_{j=1}^{k} \|\lambda_i^{1/k} \rho_i^{(j)}\|_1 = \sum_i \prod_{j=1}^{k} \lambda_i^{1/k} \|\rho_i^{(j)}\|_1$$
3. Since $\|\rho_i^{(j)}\|_1 = \operatorname{Tr}(\rho_i^{(j)}) = 1$ for all density operators, this simplifies to:
$$\|\rho\|_\pi \leq \sum_i \lambda_i = 1$$

Proof ($\Leftarrow$): $\|\rho\|_\pi \leq 1 \Longrightarrow$ separable

1. By compactness, the infimum in the norm definition is attained. So there exists a decomposition such that:
$$\rho = \sum_i Y_i^{(1)} \otimes \cdots \otimes Y_i^{(k)}, \qquad \sum_i \prod_{j=1}^{k} \|Y_i^{(j)}\|_1 = \|\rho\|_\pi \leq 1$$
2. **Key Lemma:** For any matrix A, its trace is less than or equal to its trace norm, $\operatorname{Tr}(A) \leq \|A\|_1$. Equality holds if and only if A is positive semi-definite ($A \geq 0$).
3. Now, take the trace of ρ and apply the lemma term-by-term:
$$1 = \operatorname{Tr} \rho = \sum_i \operatorname{Tr}(Y_i^{(1)}) \cdots \operatorname{Tr}(Y_i^{(k)}) \leq \sum_i \prod_{j=1}^{k} \|Y_i^{(j)}\|_1 = \|\rho\|_\pi \leq 1$$
4. This entire chain of inequalities must collapse to equality. Specifically, we must have $\sum \prod \operatorname{Tr}(Y_i^{(j)}) = \sum \prod \|Y_i^{(j)}\|_1$, which implies $\operatorname{Tr}(Y_i^{(j)}) = \|Y_i^{(j)}\|_1 \forall i, j$.
5. By the lemma, this means each factor $Y_i^{(j)}$ must be positive semi-definite.
6. We can now construct the separable form. Define $\lambda_i := \prod_j \|Y_i^{(j)}\|_1$ and $\rho_i^{(j)} := Y_i^{(j)} / \|Y_i^{(j)}\|_1$. This gives the explicit separable decomposition $\rho = \sum_i \lambda_i \bigotimes_j \rho_i^{(j)}$, completing the proof.

History: Two Worlds, One Problem

Quantum Information (1990s-2000s)

Physicists sought a definitive, computable test to distinguish separable states from entangled ones.

- **Peres (1996) - PPT Criterion:** For separable ρ, the partial transpose is positive: $\rho^{T_B} \geq 0$.
- **Rains (1999):** Used $\|\rho^{T_B}\|_1$ (the "greatest cross norm") to bound distillable entanglement.
- **Rudolph (2000-2003):** Formulated the CCN criterion, realizing it was equivalent to $\|\rho\|_\pi \leq 1$ and was an exact criterion based on the Hahn-Banach theorem.

Functional Analysis (1950s)

Grothendieck sought a canonical way to define norms on tensor product spaces to linearize bilinear maps.

- **Goal:** Given Banach spaces X, Y and a bounded bilinear map $u : X \times Y \to \mathbb{C}$, build a space $X \otimes_\pi Y$ and a linear map U such that $u(x, y) = U(x \otimes y)$ and $\|U\| = \|u\|$.
- **Solution:** He defined the **projective tensor norm** $\|\cdot\|_\pi$ to achieve this "universal linearisation".
- The separability problem was unknowingly solved decades earlier in a different context.

Definition: Tensor Norm

A tensor norm on $X \otimes Y$ is a norm $\|\cdot\|$ such that it agrees with the product of norms on simple tensors, at both the primal and dual levels:

- $\|x \otimes y\| = \|x\|_X \cdot \|y\|_Y$
- $\|\alpha \otimes \beta\|_* = \|\alpha\|_{X^*} \cdot \|\beta\|_{Y^*}$

Where $x \in X, y \in Y$ and $\alpha \in X^*, \beta \in Y^*$.

Historical Footnote

Grothendieck coined the term *reasonable cross-norm* for norms satisfying (i) and (ii). Axiom (i) makes the canonical bilinear map $j : X \times Y \to X \otimes_\alpha Y,\ (x, y) \mapsto x \otimes y$ an *isometry on elementary tensors*. Axiom (ii) ensures that the adjoint map embeds the space of rank-one bilinear forms isometrically into $(X \otimes_\alpha Y)^*$; hence every bounded bilinear form extends uniquely and continuously along j.

Universal Property

$$\begin{array}{ccc} X \times Y & \xrightarrow{j} & X \otimes_\alpha Y \\ b & \searrow \quad \swarrow & \tilde{b} \\ & \mathbb{C} & \end{array} \qquad \|j\| = 1; b \mapsto \tilde{b} \text{ is an isometry on simple tensors.}$$

Projective Norm

The **projective norm** $\|\cdot\|_\pi$ is defined by decomposition:

$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^{k} \|x_i\| \|y_i\| : z = \sum_{i=1}^{k} x_i \otimes y_i \right\}$$

Historical Footnote

Grothendieck selected $\|\cdot\|_\pi$ because it is the norm for which $(X \otimes_\pi Y)^* \simeq B(X, Y)$ holds isometrically, turning every bounded bilinear form into a bounded linear functional.

Isometry

$$\begin{array}{ccc} B(X, Y) & \xrightarrow[\text{iso}]{\cong} & (X \otimes_\pi Y)^* \\ b & \longmapsto & \tilde{b} \end{array} \qquad \|b\| = \|\tilde{b}\|.$$

Theorem: Maximal Tensor Norm

If a norm $\|\cdot\|$ on $X \otimes Y$ is a tensor norm then it is upper bounded by the projective norm.

For all $z \in X \otimes Y$:

$$\|z\| \leq \|z\|_\pi$$

Theorem: Duality & Injective Norm

The injective and projective norms are dual to each other. For Banach spaces X and Y:

- $(X \otimes_\varepsilon Y)^* = X^* \otimes_\pi Y^*$
- $(X \otimes_\pi Y)^* = X^* \otimes_\varepsilon Y^*$

Historical Footnote

The *injective norm* $\|\cdot\|_\varepsilon$ was introduced as the dual companion of $\|\cdot\|_\pi$.

Duality and bilinear maps

$$(X \otimes_\pi Y)^* \;\cong\; B(X, Y)$$
$$(X \otimes_\varepsilon Y)^* \;\cong\; L(X, Y^*)$$

Theorem: Minimal Tensor Norm

If a norm $\|\cdot\|$ on $X \otimes Y$ is a tensor norm then it is lower bounded by the injective norm.

For all $z \in X \otimes Y$:

$$\|z\|_\varepsilon \leq \|z\|$$

Theorem: Characterization

A norm $\|\cdot\|$ on $X \otimes Y$ is a tensor norm iff is sandwiched between these two norms.

$$\|z\|_\varepsilon \leq \|z\| \leq \|z\|_\pi$$

Example: Matrix Norms

The operator (spectral) norm and trace (nuclear) norm on matrices arise naturally as tensor norms.

For the space of $n \times n$ matrices $\mathcal{M}_n(\mathbb{R}) \cong \ell_2^n \otimes \ell_2^n$:

- Operator Norm (s_∞):
$(\mathcal{M}_n, \|\cdot\|_{s_\infty}) = \ell_2^n \otimes_\varepsilon \ell_2^n$
- Trace Norm (s_1):
$(\mathcal{M}_n, \|\cdot\|_{s_1}) = \ell_2^n \otimes_\pi \ell_2^n$

Historical Footnote

This identification goes back to Schatten (1950) and traces its conceptual clarity to Grothendieck's framework: the nuclear norm of a matrix is simply the projective tensor norm coming from the self-dual Hilbert space ℓ_2^n, while the spectral norm is the corresponding injective norm.

Other examples

Schatten p-norms: For $1 \leq p \leq \infty$, the Schatten p-norm on $\mathcal{M}_n$ is defined as

$$\|A\|_{s_p} = \left(\sum_{i=1}^{n} \sigma_i(A)^p \right)^{1/p}$$

where $\sigma_i(A)$ are the singular values of A.
The Schatten p-norms interpolate between the trace (nuclear) norm ($p = 1$) and the operator (spectral) norm ($p = \infty$):

$$\|A\|_{s_\infty} \leq \|A\|_{s_p} \leq \|A\|_{s_1}$$

for all $A \in \mathcal{M}_n$ and $1 \leq p \leq \infty$.